Problem 4.13

(a) Normalize R_{20} (Equation 4.82), and construct the function ψ_{200}.

(b) Normalize R_{21} (Equation 4.83), and construct ψ_{211}, ψ_{210}, and ψ_{21-1}.

Solution

The normalization of the stationary states of hydrogen requires that

$$\begin{aligned}
1 = \iiint_{\text{all space}} |\Psi_{n\ell m}(r,\theta,\phi,t)|^2\,d\mathcal{V} &= \iiint_{\text{all space}} |R(r)\Theta(\theta)\xi(\phi)T(t)|^2\,d\mathcal{V} \\
&= \iiint_{\text{all space}} \left|R_{n\ell}(r)Y_\ell^m(\theta,\phi)e^{-iE_nt/\hbar}\right|^2 d\mathcal{V} \\
&= \int_0^\pi \int_0^{2\pi} \int_0^\infty |R_{n\ell}(r)|^2|Y_\ell^m(\theta,\phi)|^2(r^2\sin\theta\,dr\,d\phi\,d\theta) \\
&= \underbrace{\left[\int_0^\infty r^2|R_{n\ell}(r)|^2\,dr\right]}_{=1}\underbrace{\left[\int_0^\pi \int_0^{2\pi}|Y_\ell^m(\theta,\phi)|^2\sin\theta\,d\phi\,d\theta\right]}_{=1}.
\end{aligned}$$

Part (a)

Equation 4.82 is on page 148.

$$R_{20}(r) = \frac{c_0}{2a}\left(1-\frac{r}{2a}\right)e^{-r/2a} \tag{4.82}$$

So then

$$\begin{aligned}
1 &= \int_0^\infty r^2|R_{20}(r)|^2\,dr \\
&= \int_0^\infty r^2\left[\frac{c_0}{2a}\left(1-\frac{r}{2a}\right)e^{-r/2a}\right]^2 dr \\
&= \frac{c_0^2}{4a^2}\int_0^\infty r^2\left(1-\frac{r}{2a}\right)^2 e^{-r/a}\,dr \\
&= \frac{c_0^2}{4a^2}\int_0^\infty r^2\left(1-\frac{r}{a}+\frac{r^2}{4a^2}\right)e^{-r/a}\,dr \\
&= \frac{c_0^2}{4a^2}\int_0^\infty \left(r^2-\frac{r^3}{a}+\frac{r^4}{4a^2}\right)e^{-r/a}\,dr \\
&= \frac{c_0^2}{4a^2}\left(\int_0^\infty r^2e^{-r/a}\,dr - \frac{1}{a}\int_0^\infty r^3e^{-r/a}\,dr + \frac{1}{4a^2}\int_0^\infty r^4e^{-r/a}\,dr\right) \\
&= \frac{c_0^2}{4a^2}\left[\int_0^\infty \frac{\partial^2}{\partial u^2}(e^{-ur})\bigg|_{u=\frac{1}{a}}dr + \frac{1}{a}\int_0^\infty \frac{\partial^3}{\partial u^3}(e^{-ur})\bigg|_{u=\frac{1}{a}}dr + \frac{1}{4a^2}\int_0^\infty \frac{\partial^4}{\partial u^4}(e^{-ur})\bigg|_{u=\frac{1}{a}}dr\right].
\end{aligned}$$

Pull the derivatives outside the integrals.

$$
\begin{aligned}
1 &= \frac{c_0^2}{4a^2}\left[\frac{d^2}{du^2}\left(\int_0^\infty e^{-ur}\,dr\right)\bigg|_{u=\frac{1}{a}} + \frac{1}{a}\frac{d^3}{du^3}\left(\int_0^\infty e^{-ur}\,dr\right)\bigg|_{u=\frac{1}{a}} + \frac{1}{4a^2}\frac{d^4}{du^4}\left(\int_0^\infty e^{-ur}\,dr\right)\bigg|_{u=\frac{1}{a}}\right] \\
&= \frac{c_0^2}{4a^2}\left[\frac{d^2}{du^2}\left(-\frac{1}{u}e^{-ur}\bigg|_0^\infty\right)\bigg|_{u=\frac{1}{a}} + \frac{1}{a}\frac{d^3}{du^3}\left(-\frac{1}{u}e^{-ur}\bigg|_0^\infty\right)\bigg|_{u=\frac{1}{a}} + \frac{1}{4a^2}\frac{d^4}{du^4}\left(-\frac{1}{u}e^{-ur}\bigg|_0^\infty\right)\bigg|_{u=\frac{1}{a}}\right] \\
&= \frac{c_0^2}{4a^2}\left[\frac{d^2}{du^2}\left(\frac{1}{u}\right)\bigg|_{u=\frac{1}{a}} + \frac{1}{a}\frac{d^3}{du^3}\left(\frac{1}{u}\right)\bigg|_{u=\frac{1}{a}} + \frac{1}{4a^2}\frac{d^4}{du^4}\left(\frac{1}{u}\right)\bigg|_{u=\frac{1}{a}}\right] \\
&= \frac{c_0^2}{4a^2}\left[\frac{d}{du}\left(-\frac{1}{u^2}\right)\bigg|_{u=\frac{1}{a}} + \frac{1}{a}\frac{d^2}{du^2}\left(-\frac{1}{u^2}\right)\bigg|_{u=\frac{1}{a}} + \frac{1}{4a^2}\frac{d^3}{du^3}\left(-\frac{1}{u^2}\right)\bigg|_{u=\frac{1}{a}}\right] \\
&= \frac{c_0^2}{4a^2}\left[\left(\frac{2}{u^3}\right)\bigg|_{u=\frac{1}{a}} + \frac{1}{a}\frac{d}{du}\left(\frac{2}{u^3}\right)\bigg|_{u=\frac{1}{a}} + \frac{1}{4a^2}\frac{d^2}{du^2}\left(\frac{2}{u^3}\right)\bigg|_{u=\frac{1}{a}}\right] \\
&= \frac{c_0^2}{4a^2}\left[\left(\frac{2}{u^3}\right)\bigg|_{u=\frac{1}{a}} + \frac{1}{a}\left(-\frac{6}{u^4}\right)\bigg|_{u=\frac{1}{a}} + \frac{1}{4a^2}\frac{d}{du}\left(-\frac{6}{u^4}\right)\bigg|_{u=\frac{1}{a}}\right] \\
&= \frac{c_0^2}{4a^2}\left[\left(\frac{2}{u^3}\right)\bigg|_{u=\frac{1}{a}} + \frac{1}{a}\left(-\frac{6}{u^4}\right)\bigg|_{u=\frac{1}{a}} + \frac{1}{4a^2}\left(\frac{24}{u^5}\right)\bigg|_{u=\frac{1}{a}}\right] \\
&= \frac{c_0^2}{4a^2}\left[(2a^3) + \frac{1}{a}(-6a^4) + \frac{1}{4a^2}(24a^5)\right] \\
&= \frac{c_0^2}{4a^2}(2a^3 - 6a^3 + 6a^3) \\
&= \frac{a}{2}c_0^2
\end{aligned}
$$

Solve for c_0.

$$c_0 = \sqrt{\frac{2}{a}}$$

Therefore, the normalized radial eigenfunction is

$$R_{20}(r) = \frac{c_0}{2a}\left(1 - \frac{r}{2a}\right)e^{-r/2a} = \sqrt{\frac{1}{2a^3}}\left(1 - \frac{r}{2a}\right)e^{-r/2a},$$

and the spatial part of the wave function is

$$
\begin{aligned}
\psi_{200}(r,\theta,\phi) &= R_{20}(r)Y_0^0(\theta,\phi) \\
&= \left[\sqrt{\frac{1}{2a^3}}\left(1 - \frac{r}{2a}\right)e^{-r/2a}\right]\left(\sqrt{\frac{1}{4\pi}}\right) \\
&= \sqrt{\frac{1}{8\pi a^3}}\left(1 - \frac{r}{2a}\right)e^{-r/2a}.
\end{aligned}
$$

Part (b)

Equation 4.83 is on page 148.

$$R_{21}(r) = \frac{c_0}{4a^2} r e^{-r/2a} \tag{4.83}$$

So then

$$\begin{aligned}
1 &= \int_0^\infty r^2 |R_{21}(r)|^2 \, dr \\
&= \int_0^\infty r^2 \left[\frac{c_0}{4a^2} r e^{-r/2a} \right]^2 dr \\
&= \int_0^\infty r^2 \left[\frac{c_0^2}{16a^4} r^2 e^{-r/a} \right] dr \\
&= \frac{c_0^2}{16a^4} \int_0^\infty r^4 e^{-r/a} \, dr \\
&= \frac{c_0^2}{16a^4} \int_0^\infty \left. \frac{\partial^4}{\partial u^4}(e^{-ur}) \right|_{u=\frac{1}{a}} dr \\
&= \frac{c_0^2}{16a^4} \frac{d^4}{du^4} \left. \left(\int_0^\infty e^{-ur} \, dr \right) \right|_{u=\frac{1}{a}} \\
&= \frac{c_0^2}{16a^4} \frac{d^4}{du^4} \left. \left(-\frac{1}{u} e^{-ur} \Big|_0^\infty \right) \right|_{u=\frac{1}{a}} \\
&= \frac{c_0^2}{16a^4} \frac{d^4}{du^4} \left. \left(\frac{1}{u} \right) \right|_{u=\frac{1}{a}} \\
&= \frac{c_0^2}{16a^4} \frac{d^3}{du^3} \left. \left(-\frac{1}{u^2} \right) \right|_{u=\frac{1}{a}} \\
&= \frac{c_0^2}{16a^4} \frac{d^2}{du^2} \left. \left(\frac{2}{u^3} \right) \right|_{u=\frac{1}{a}} \\
&= \frac{c_0^2}{16a^4} \frac{d}{du} \left. \left(-\frac{6}{u^4} \right) \right|_{u=\frac{1}{a}} \\
&= \frac{c_0^2}{16a^4} \left. \left(\frac{24}{u^5} \right) \right|_{u=\frac{1}{a}} \\
&= \frac{c_0^2}{16a^4} (24a^5) \\
&= \frac{3a}{2} c_0^2.
\end{aligned}$$

Solve for c_0.

$$c_0 = \sqrt{\frac{2}{3a}}$$

Therefore, the normalized radial eigenfunction is

$$\begin{aligned} R_{21}(r) &= \frac{c_0}{4a^2} r e^{-r/2a} \\ &= \sqrt{\frac{2}{3a}} \frac{1}{4a^2} r e^{-r/2a} \\ &= \sqrt{\frac{1}{24a^5}}\, r e^{-r/2a}, \end{aligned}$$

and the spatial part of each of the three wave functions is

$$\begin{aligned} \psi_{211}(r,\theta,\phi) &= R_{21}(r) Y_1^1(\theta,\phi) \\ &= \left(\sqrt{\frac{1}{24a^5}}\, r e^{-r/2a}\right)\left(-\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{i\phi}\right) \\ &= -\sqrt{\frac{1}{64\pi a^5}}\, r e^{-r/2a} \sin\theta\, e^{i\phi} \end{aligned}$$

and

$$\begin{aligned} \psi_{210}(r,\theta,\phi) &= R_{21}(r) Y_1^0(\theta,\phi) \\ &= \left(\sqrt{\frac{1}{24a^5}}\, r e^{-r/2a}\right)\left(\sqrt{\frac{3}{4\pi}} \cos\theta\right) \\ &= \sqrt{\frac{1}{32\pi a^5}}\, r e^{-r/2a} \cos\theta \end{aligned}$$

and

$$\begin{aligned} \psi_{21-1}(r,\theta,\phi) &= R_{21}(r) Y_1^{-1}(\theta,\phi) \\ &= \left(\sqrt{\frac{1}{24a^5}}\, r e^{-r/2a}\right)\left(\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{-i\phi}\right) \\ &= \sqrt{\frac{1}{64\pi a^5}}\, r e^{-r/2a} \sin\theta\, e^{-i\phi}. \end{aligned}$$